Computation and proof of explicit continued fractions

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A long history

- Huygens' planetarium [Huygens, 1682]
- De analysis infinitorum [Euler, 1748]
- π is irrational [Lambert, 1761]
- real roots isolation of $P \in \mathbb{R}[X]$ [Lenstra, 2002, Hallgren, 2007]
- *A new Stirling series as continued fraction* [C. Mortici, 2009]

Renewed interest for numerical evaluation

- *Handbook of mathematical functions* [Abramowitz and Stegun, 1964]
- *Continued Fractions for numerical analysis* [Jones and Thron, 1988]
- *Continued Fractions with applications* [Lorentzen and Waadeland, 1992]
- *Handbook of Continued Fractions for Special Functions*
  [Cuyt, Peterson, Verdonk, Waadeland and Jones, 2008]

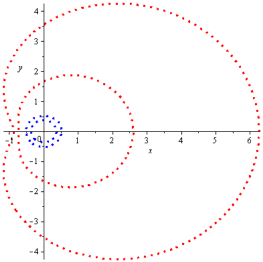
Convergence

$$\ln(1+x) = \sum_{1}^{4}(-1)^{i+1}\frac{x^i}{i} + O(x^5)$$

$$= \frac{1/2\,x^2 + x}{1/6\,x^2 + x + 1} + O(x^5)$$

$$= \frac{x}{1 + \frac{x/2}{1 + \frac{x/6}{1 + x/3}}} + O(x^5)$$

zone with 10 correct digits, for $x \in \mathbb{C}$ (with $n = 20$ and $n = 30$)



- Taylor at order $2n$: convergence for $|x| < 1$,
- Padé et order n/n : convergence for $1 + x \notin \mathbb{R}_-$.

Corresponding continued fractions

(truncated) *C-fraction*:

$$\mathop{\mathrm{K}}_{i=0}^{n} \frac{a_i x^{\alpha_i}}{1} := \cfrac{a_0 x^{\alpha_0}}{1 + \cfrac{a_1 x^{\alpha_1}}{1 + \cfrac{\cdots}{1 + \cfrac{a_{n-1} x^{\alpha_{n-1}}}{1 + a_n x^{\alpha_n}}}}}, \quad a_i \in \mathbb{C}^\star,\ \alpha_i \in \mathbb{N}^*$$

Correspondence:

$$\left\{ \mathop{\mathrm{K}}_{i=0}^{\infty} \frac{a_i x^{\alpha_i}}{1} \right\} \simeq \left\{ \sum_{i=1}^{\infty} c_i x^i \right\}$$

$$\left(\begin{bmatrix} a_0 \\ \alpha_0 \end{bmatrix}, \ldots, \begin{bmatrix} a_n \\ \alpha_n \end{bmatrix} \right) \overset{\text{straightforward}}{\leftrightsquigarrow} (c_1, \ldots, c_{\alpha_0 + \ldots + \alpha_n})$$

regular C-fractions : when $\alpha_i = 1$.

simple, many formulas and applications. [Stieltjes 1894, Cuyt *et alii* 2008]

Formulas

Some general classes

- [Gauss]

$$\frac{{}_2F_1(a,b;c;z)}{{}_2F_1(a,b+1;c+1;z)} = 1 + \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{p_m z}{1}$$

$$p_{2k} = -\frac{(k+b)(k+c-a)}{(2k+c)(2k-1+c)}, p_{2k+1} = -\frac{(k+a)(k+c-b)}{(2k+c)(2k+1+c)}$$

where ${}_2F_1(a,b;c;z) := \sum_{n\geq 0} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$, and $(a)_n := a(a+1)\cdots(a+n-1)$.

- [Khovanskii 1963] similar solution for

$$(1+\alpha z)zy' + (\beta+\gamma z)y + \delta y^2 = \epsilon z,\ y(0) = -\beta/\gamma.$$

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$\rightsquigarrow$ We look for C-fractions $\mathcal{K}_{m=1}^{\infty} \frac{a_m z}{1}$ with a_{2k} and a_{2k+1} rational in k.

Proofs

Proofs are performed:

- by generalization/specialization,
- sometimes indirectly, as in:

$$\exp(x) = \lim_{k\to\infty} \left(1 + \frac{x}{k}\right)^k.$$

They rely on:

- 3 terms linear recurrences,
- the invariance of Riccati equations with rational coefficients

$$y'(z) = py^2 + qy + r, \qquad p, q, r \in \mathbb{C}(z),$$

  under Mœbius transforms $y_2(z) := \dfrac{az}{1 + y(z)}$ with $a \in \mathbb{C}^\star$,
- and *q-analogues* of these.

The equation as data structure

Proposition (Cauchy): the Riccati equation $y'(z) = py^2 + qy + r$ with $y(0) = 0$ and $p, q, r \in \mathbb{C}(z)$ admits a unique power series solution y_0. A sequence of power series $(f_n)_{n \geq 0}$ tends to y_0 *iff* the remainder valuations satisfy:

$$\mathrm{val}(f_n' - pf_n^2 - qf_n - r) \to \infty$$

where $\mathrm{val}\left(\sum_{i \geq 0} c_i z^i\right) := \min\{i \geq 0 \mid c_i \neq 0\}$.

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Aim: *one* procedure

$$\{y'(z) = py^2 + qy + r,\ y(0) = 0\} \quad \mapsto \quad \text{explicit C-fraction} + \text{proof}$$

- two steps: *guessing* the formula, and *proving* it,
- the equation is non-linear,
- classical tools are linear.

A *guess and prove* approach

`riccati_to_cfrac(`$\{\exp'(x)=\exp(x), \exp(0)=1\}, x=0$`);`

$$\exp(x) = 1 + \cfrac{x}{1 + \cfrac{\cdots}{1 + \cfrac{\frac{1}{2(2k+1)}x}{1 + \cfrac{\frac{-1}{2(2k+1)}x}{\cdots}}}}$$

- Direct computation : $(a_0,\dots,a_{19})=(1,\frac{-1}{2},\frac{1}{6},\frac{-1}{6},\dots,\frac{1}{10},\frac{-1}{10},\dots,\frac{1}{38},\frac{-1}{38})$.
- Linear algebra : $\{-n^2a_n+na_{n+1}+n(n+3)a_{n+2}=0,\quad (a_0,a_1,a_2)=(1,-\frac{1}{2},\frac{1}{6})\}$,

$$\rightsquigarrow\quad a_{2k}=\frac{1}{2(2k+1)},\qquad a_{2k+1}=\frac{-1}{2(2k+1)},\qquad k>0.$$

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  $\rightsquigarrow \quad a_{2k}=\frac{1}{2(2k+1)}, \qquad a_{2k+1}=\frac{-1}{2(2k+1)}, \qquad k>0.$
- Proof : $f_n := 1 + \mathcal{K}_{i=0}^{n} \frac{a_i x}{1} \underset{n\to\infty}{\longrightarrow} \exp(x)?$

. . .

D-finiteness

A sequence is *D-finite* when it satisfies a linear recurrence with polynomial coefficients. This recurrence is an effective data structure:

- $(a_n)_{n\geq 0} = 0$ is decidable;
- a recurrence can be computed for:
  $(a_{n+1})_{n\geq 0}$, $(a_n - b_n)_{n\geq 0}$, $(a_n b_n)_{n\geq 0}$, $\left(\sum_{i+j=n} a_i b_j\right)_{n\geq 0}$ ...

Example (squaring)

```
> rec:={ u(n+2) = (n+1)*u(n+1) + 2*u(n),  u(0)=0, u(1)=1 }:
> gfun:-poltorec( u(n)^2, [rec], [u(n)], c(n) );
```

$$\Big\{\left(8\,n+16\right)c\left(n\right)+\left(-2\,n^3-8\,n^2-14\,n-8\right)c\left(n+1\right)+\left(-n^3-5\,n^2-10\,n-8\right)c\left(n+2\right)+\left(n+1\right)c\left(n+3\right),\quad c\left(0\right)=0, c\left(1\right)=1, c\left(2\right)=1\Big\}$$

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$\rightsquigarrow$ *3 generators over* $\mathbb{Q}(n)$ *:* $u(n)^2$, $u(n)u(n+1)$, $u(n+1)^2$

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D-finiteness of the remainder

Recall that we wanted: $\mathrm{val}(f_n' - pf_n^2 - qf_n - r) \to \infty$.

for $f_n = 1 + \cfrac{a_0 x}{1 + \cfrac{\cdots}{1 + \cfrac{a_n x}{1}}}$ and p, q, r rational, we have $f_n = P_n / Q_n$ with

$$\begin{cases} P_n = P_{n-1} + a_n x\, P_{n-2} & (P_{-2}, P_{-1}) = (1, 0) \\ Q_n = Q_{n-1} + a_n x\, Q_{n-2} & (Q_{-2}, Q_{-1}) = (0, 1) \end{cases}$$

Lemma

Let $H_n := Q_n^2(f_n' - pf_n^2 - qf_n - r)$, *then* $f_n \to y_0(z) \iff \mathrm{val}(H_{2n}) \to \infty$

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- $H_{n+i} \in \mathrm{Vect}(a_n P_n' Q_n,\ \ a_n P_n Q_n',\ \ P_{n+1} Q_{n+1}',\ \ \ldots)$ of finite dimension !
- $\leadsto$ first recurrence for H_{2n},
- Does not conclude $\Longrightarrow$ reduction of order . . . DEMO

Principle: Guess and prove again !

Theorem (Dimension of the solutions space)

$\forall n,\ p_0(n) \neq 0 \implies \dim\{(u_n)_{n\geq 0},\quad p_0(n)u_{n+d} + \cdots + p_d(n)u_n = 0\} = d.$

Problem (2^{-n} et 2^n sont dans un bateau...)

Given $\{2a_{n+2} - 5a_{n+1} + 2a_n = 0, (a_0, a_1) = (1, \frac{1}{2})\}$,
show that $\lim_{n\to\infty} a_n = 0$.

- *guess* a "small" recurrence: $a_{n+1} + \alpha a_n = 0$?
  $(a_0, a_1) \rightsquigarrow \alpha = -\frac{1}{2}$. $\{2b_{n+1} - b_n = 0, b_0 = 1\}$
- it defines a *new* sequence $(b_n)_{n\geq 0}$,
- we prove $b_n = a_n$, by induction:
  - $(b_0, b_1) = (a_0, a_1)$;
  - $2b_{n+2} - 5b_{n+1} + 2b_n = (2b_{n+2} - b_{n+1}) - 2(2b_{n+1} - b_n) = 0$.

Miracle

- all Riccati solutions in [Cuyt *et alii*, 2008] are covered,
- and give a hypergeometric $(H_n)_{n\geq 0}$!
- it generalizes to difference, and q-difference equations,

  $\Longrightarrow$ all explicit formulas in [Cuyt *et alii*, 2008]

Question: reciprocal ?

Classification

What are the

- regular C-fractions
- solutions of a Riccati equation with rational coefficients
- with a hypergeometric remainder H_{2n} ?

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. . . the formula of Khovanskii [Khovanskii 1963] for

$$(1+\alpha z)zy' + (\beta+\gamma z)y + \delta y^2 = \epsilon z, \; y(0) = -\beta/\gamma$$

is the most general one ! (has a caracterization)

Conclusion

- One procedure,
  - Riccati equation with rational coefficients
    $\mapsto$ C-fraction with rational coefficients,
  - proof with a quick and direct computation,
- an implementation
  - prototype `gfun:-ContFrac`
  - generalizes by hand (to difference and q-difference equations),
- and a classification. . .
  - to be continued