Formal Proof of Cody & Waite's Exponential

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Approximating the Exponential with FP Numbers

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Correctness condition: the relative error between `cw_exp(x)` and the mathematical value $\exp x$ is less than 2^{-51}.

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Algorithm Overview and Error Analysis

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&= \exp t \cdot \exp(-\varepsilon_t) \cdot 2^k \quad \text{with } t \simeq x - k \cdot \log 2 \\
&= f(t) \cdot (1 + \varepsilon_f)^{-1} \cdot \exp(-\varepsilon_t) \cdot 2^k \quad \text{with } f \simeq \exp
\end{aligned}
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&= f(t) \cdot (1 + \varepsilon_f)^{-1} \cdot \exp(-\varepsilon_t) \cdot 2^k \quad \text{with } f \simeq \exp \\
&= \tilde{f}(t) \cdot (1 + \varepsilon_{\tilde{f}})^{-1} \cdot (1 + \varepsilon_f)^{-1} \cdot \exp(-\varepsilon_t) \cdot 2^k
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So $\tilde{f}(t) \cdot 2^k$ approximates $\exp x$ with a relative error $\approx \varepsilon_{\tilde{f}} + \varepsilon_f + \varepsilon_t$.

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So $\tilde{f}(t) \cdot 2^k$ approximates $\exp x$ with a relative error $\approx \varepsilon_{\tilde{f}} + \varepsilon_f + \varepsilon_t$.

Goal: design the function and bound the following expressions

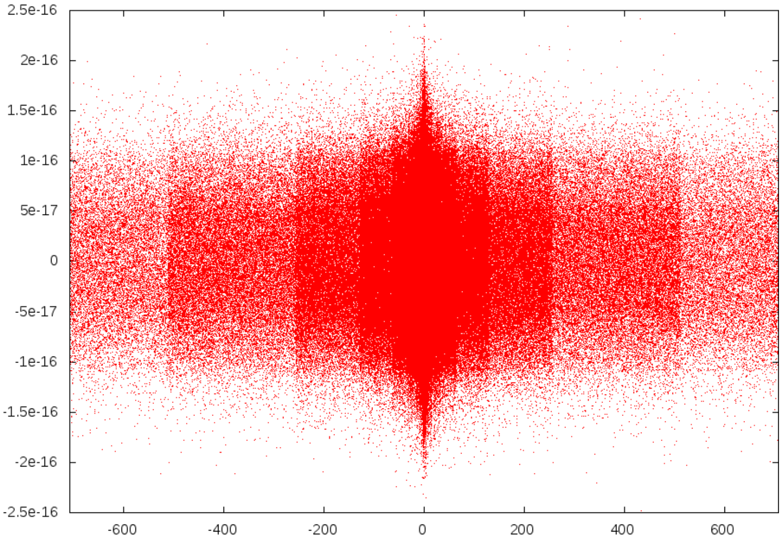
- reduced argument t, (f depends on the range of t)
- argument reduction error $\varepsilon_t = t - (x - k \cdot \log 2)$,
- relative method error $\varepsilon_f = f(t) / \exp t - 1$,
- relative round-off error $\varepsilon_{\tilde{f}} = \tilde{f}(t) / f(t) - 1$.

C Implementation

```
double cw_exp(double x)
{
  if (fabs(x) > 710.) return x < 0. ? 0. : INFINITY;
  double Log2h = 0xb.17217f7d1c00p-4;
  double Log2l = 0xf.79abc9e3b398p-48;
  double InvLog2 = 0x1.71547652b82fep0;
  double p1 = 0x1.c70e46fb3f692p-8;
  double p2 = 0x1.152a46f58dc1cp-16;
  double q1 = 0xe.38c738a128d98p-8;
  double q2 = 0x2.07f32dfbc7012p-12;

  double k = nearbyint(x * InvLog2);
  double t = x - k * Log2h - k * Log2l;
  double t2 = t * t;
  double p = 0.25 + t2 * (p1 + t2 * p2);
  double q = 0.5  + t2 * (q1 + t2 * q2);
  double f = t * (p / (q - t * p)) + 0.5;
  return ldexp(f, (int)k + 1);
}
```

Total Relative Error



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How do you automate proofs on real and FP numbers?

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Use interval arithmetic.

Outline

1. Cody & Waite's exponential
2. Formalizing floating-point algorithms: Flocq
   - Tool description
   - Specification of the algorithm
   - Proof structure
3. Bounding method errors: Coq.Interval
4. Bounding round-off errors: Gappa
5. Conclusion

Flocq: a Floating-Point Formalization for Coq

Support

- multi-radix (2, 10, exotic),
- multi-format (fixed-point, floating-point, exotic).
- axiomatic rounding operators (no overflow),
- computable IEEE-754 operators, including $\div$ and $\sqrt{\cdot}$,
- comprehensive library of generic theorems.

Coq Implementation and Correctness Property

Flocq-based description

```
Definition cw_exp (x : R) :=
  let k := nearbyint (mul x InvLog2) in
  let t := sub (sub x (mul k Log2h)) (mul k Log2l) in
  let t2:= mul t t in
  let p := add p0 (mul t2 (add p1 (mul t2 p2))) in
  let q := add q0 (mul t2 (add q1 (mul t2 q2))) in
  let f:= add (mul t (div p (sub q (mul t p)))) 1/2 in
  pow2 (Zfloor k + 1) * f.

Theorem exp_correct :
  forall x : R,
  generic_format radix2 (FLT_exp (-1074) 53) x ->
  Rabs x <= 710 ->
  Rabs ((cw_exp x - exp x) / exp x) <= 1 * pow2 (-51).
```

Intermediate Lemmas

```
Lemma method_error :
  forall t : R,
  let t2 := t * t in
  let p := p0 + t2 * (p1 + t2 * p2) in
  let q := q0 + t2 * (q1 + t2 * q2) in
  let f := 2 * (t * (p / (q - t * p)) + 1/2) in
  Rabs t <= 355 / 1024 ->
  Rabs ((f - exp t) / exp t) <= 23 * pow2 (-62).

Lemma argument_reduction :
  forall x : R,
  generic_format radix2 (FLT_exp (-1074) 53) x ->
  Rabs x <= 710 ->
  let k := nearbyint (mul x InvLog2) in
  let t := sub (sub x (mul k Log2h)) (mul k Log2l) in
  Rabs t <= 355 / 1024 /\
  Rabs (t - (x - k * ln 2)) <= 65537 * pow2 (-71).
```

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   - Tool description
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Intermediate Lemma: Method Error

```
Lemma method_error :
  forall t : R,
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  let p := p0 + t2 * (p1 + t2 * p2) in
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  Rabs t <= 355 / 1024 ->
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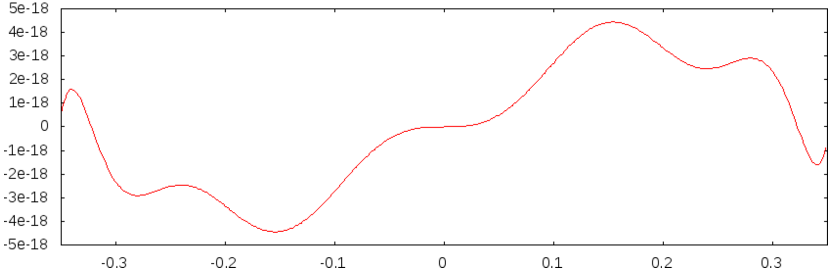

Automatic Proof using Coq.Interval

Support

Quantifier-free formulas of enclosures of expressions using

- basic arithmetic operators: $+$, $-$, $\times$, $\div$, $\sqrt{\cdot}$,
- elementary functions: cos, sin, tan, arctan, exp, log.

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Approach

Fully formalized in Coq:

- efficient multi-precision FP arithmetic,
- interval arithmetic with univariate Taylor models,
- reflexive tactic.

Bounding Errors Automatically

Naive interval arithmetic cannot compute tight bounds for

$$\frac{f(t) - \exp t}{\exp t} \in \frac{[0.7, 1.5] - [0.7, 1.5]}{[0.7, 1.5]} = \frac{[-0.8, 0.8]}{[0.7, 1.5]} \subseteq [-1.2, 1.2]$$

due to the dependency effect.

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due to the dependency effect.

But one can automatically compute a polynomial P and an interval Δ such that

$$\frac{f(t) - \exp t}{\exp t} = P(t) + \delta(t) \quad \text{with } \delta(t) \in \Delta$$

and then use naive interval arithmetic to compute tight bounds for

$$P(t) + \delta(t) \in [-23 \cdot 2^{-62}, 23 \cdot 2^{-62}].$$

Outline

Relative Round-off Error

```
double cw_exp(double x) {
  ...
  //@ assert \abs(t) <= 355. / 1024.;
  double t2 = t * t;
  double p = 0.25 + t2 * (p1 + t2 * p2);
  double q = 0.5  + t2 * (q1 + t2 * q2);
  double f = t * (p / (q - t * p)) + 0.5;
  //@ assert \abs((f - \exp(t)) / \exp(t)) <= ...;
  ...
}
```

Automatic Proof using Gappa

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- binary floating-/fixed-point rounding operators,
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1. symbolic proof search of relevant theorems,
2. numerical application of selected instances,
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Database of ≈ 150 theorems

- naive interval arithmetic,
- rewriting of errors between structurally-similar expressions.

Errors Between Structurally-similar Expressions

Let us suppose that $\tilde{u}$ and u are close, and $\tilde{v}$ and v too.
How to bound

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But it works by rewriting

$$\tilde{u} \cdot \tilde{v} - u \cdot v = (\tilde{u} - u) + (\tilde{v} - v) + (\tilde{u} - u) \cdot (\tilde{v} - v)$$

and then by naive interval arithmetic.

Bounding the Relative Round-off Error using Gappa

First Try

```
t2 double= t * t;
p double= 0.25 + t2 * (p1 + t2 * p2);
q double= 0.5  + t2 * (q1 + t2 * q2);
f double= t * (p / (q - t * p)) + 0.5;

Mt2 = t * t;
Mp = 0.25 + Mt2 * (p1 + Mt2 * p2);
Mq = 0.5  + Mt2 * (q1 + Mt2 * q2);
Mf = t * (Mp / (Mq - t * Mp)) + 0.5;

{ |t| <= 355b-10 -> f -/ Mf in ? }
```

Argument Reduction

How to compute $x - k \cdot \log 2$?

Naive implementation

```
double k = nearbyint(x * 0x1.71547652b82fep0);
double t = x - k * 0xb.17217f7d1cf78p-4;
```

For $x = 700$, we get $k = 1010$ and $\varepsilon_t \simeq 2^{-44.2}$.

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For $x = 700$, we get $k = 1010$ and $\varepsilon_t \simeq 2^{-44.2}$.

Cody & Waite's trick

```
double k = nearbyint(x * 0x1.71547652b82fep0);
double Log2h = 0xb.17217f7d1cp-4; // 42 bits out of 53
double Log2l = 0xf.79abc9e3b398p-48;
double t = (x - k * Log2h) - k * Log2l;
```

For $x = 700$, we get $k = 1010$ and $\varepsilon_t \simeq 2^{-58.1}$.

Bounding Errors Automatically (1/2)

Gappa cannot compute tight bounds for

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due to the dependency effect inherent to interval arithmetic.

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But it can compute tight bounds for

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since it is an error between two structurally-similar expressions.

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User hint: $x = (x \cdot \texttt{InvLog2}) \cdot \texttt{InvLog2}^{-1}$.

Bounding Errors Automatically (2/2)

Gappa cannot compute tight bounds for

$$((x - k \cdot \texttt{Log2h}) - k \cdot \texttt{Log2l}) - (x - k \cdot \log 2)$$

due to the dependency effect and the use of log.

Bounding Errors Automatically (2/2)

Gappa cannot compute tight bounds for

$$((x - k \cdot \texttt{Log2h}) - k \cdot \texttt{Log2l}) - (x - k \cdot \log 2)$$

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But it can compute tight bounds for

$$((x - k \cdot \texttt{Log2h}) - k \cdot \texttt{Log2l}) - ((x - k \cdot \texttt{Log2h}) - k \cdot \mu)$$

since it is an error between two structurally-similar expressions, as long as the user gives some bounds on $\mu = \log 2 - \texttt{Log2h}$.

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since it is an error between two structurally-similar expressions, as long as the user gives some bounds on $\mu = \log 2 - \texttt{Log2h}$.

User hints: $x - k \cdot \log 2 = x - k \cdot \texttt{Log2h} - k \cdot (\log 2 - \texttt{Log2h})$ and $\texttt{Log2l} - (\log 2 - \texttt{Log2h}) \in [-2^{-102}, 0]$.

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Proof Summary

- Relative method error:
  - multi-precision interval arithmetic using Taylor models,
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- Argument reduction (tricky code):
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  - partly automated proof, user interactions:
    - a case analysis for excluding $x \simeq 0$,
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  - partly automated proof, user interactions:
    - a case analysis for excluding $x \simeq 0$,
    - two trivial identities, (developer knowledge)
    - some bounds on $\log 2$ using interval arithmetic.
- Result reconstruction and total error:
  - straightforward manual proof + interval arithmetic.

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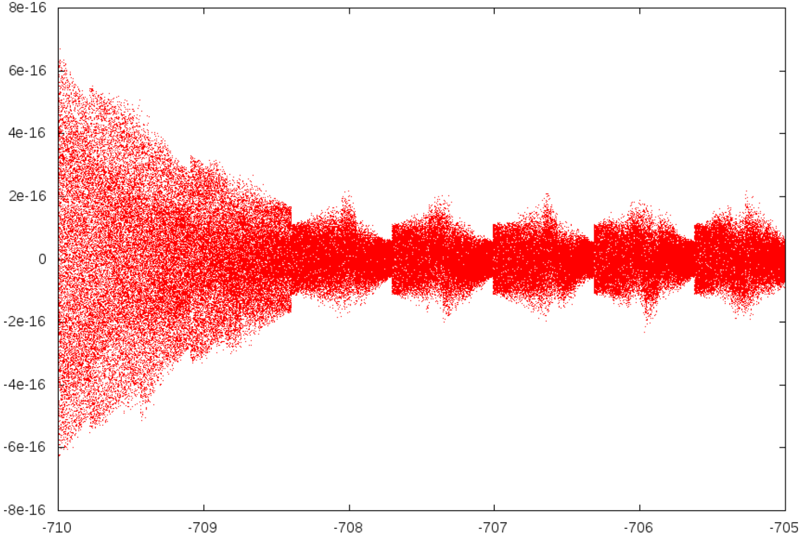
The theorem is formally proved, but what does it actually state?

- Flocq's abstract formats have no upper bound
  $\Rightarrow$ need for additional proofs to ensure no overflow occurs.
- Result reconstruction is proved over real numbers
  $\Rightarrow$ on subnormal numbers, the relative error explodes.

Total Relative Error (Subnormal Results)



Questions?

Thanks to dedicated automations, formally proving the correctness of floating-point algorithms is now accessible to non-specialists.

Flocq:	`http://flocq.gforge.inria.fr/`
Gappa:	`http://gappa.gforge.inria.fr/`
Coq.Interval:	`http://coq-interval.gforge.inria.fr/`