Bernstein Coefficients

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May 2015

Basic idea

- Proving positivity of a polynomial in a bounded interval
- two points of view
  - Variable changes to change the nature of intervals
  - Basis change in the polynomial vector space
- Obtain a sufficient positivity condition on the coefficients
- A natural approach for eliminating a variable
- Condition is not necessary
  - incompleteness is solvable thanks to dichotomy

Variable changes

- Positivity of P on interval (a, b)
  equivalent to positivity of $P \circ (X * (b - a) + a)$ on interval $(0, 1)$
- Positivity of P on interval $(a, +\infty)$
  equivalent to positivity of $P \circ (X + a)$ on $(0, +\infty)$
- Positivity of P on $(0, 1)$
  equivalent to positivity of $X^n \times P(1/X)$ on $(1, +\infty)$
  - Last operation actually stays in the same polynomial space (if $deg(P) = n$)
  - Alternative point of view: *reversing the list coefficient*
  - A linear operation

Linear transforms

- for any Q, $P \mapsto P \circ Q$ is linear
- call $\theta_a : P \mapsto P \circ (X + a)$
- call $\chi_a : P \mapsto P \circ (aX)$
- call $\rho_n : \sum_{i=0}^{n} c_i X^i \mapsto \sum_{i=0}^{n} c_{n-i} X^i$
- positivity of P on interval (a, b)
  equivalent to positivity of $\theta_a \circ \chi_{b-a}(P)$ on $(0, 1)$
- positivity of P on interval $(0, 1)$
  equivalent to positivity of $\theta_1 \circ \rho_n(P)$ on $(0, +\infty)$

Basis change

- the operation $\mu_{n,a,b} = \theta_1 \circ \rho_n \circ \theta_a \circ \chi_{b-a}$ is linear, invertible
  - On vector space of polynomials of degree $\leq n$
- Coefficients of $\mu_{n,a,b}(P)$ in monomial basis $(1, X, \ldots, X^n)$ are coefficients of P in basis $\mu^{-1}_{n,a,b}(1, X, \ldots, X^n)$

$$\mu_{n,a,b}(P) = \sum_{i=0}^{n} b_i X^i \Leftrightarrow P = \sum_{i=0}^{n} b_i \mu^{-1}_{n,a,b}(X^i)$$

- $\forall i, 0 < b_i \Rightarrow \forall x, a < x < b \Rightarrow 0 < P(x)$
- $\mu^{-1}_{n,a,b}(X^k) = \frac{(X-a)^{n-k}(b-X)^k}{(b-a)^n}$
- The polynomials $\mu^{-1}_{n,a,b}(X^k)$ are obviously positive on (a, b)

Proof procedure

- Use $\mu_{n,a,b}$ to compute coefficients b_i
- Verify the equality

$$P = \sum_{i=0}^{n} b_i \frac{(X-a)^{n-k}(b-X)^k}{(b-a)^n}$$

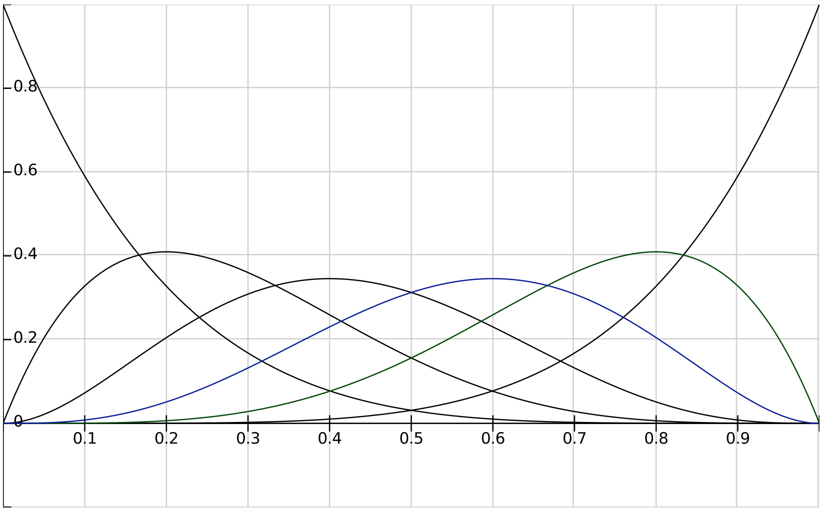
- get rid of null coefficients, and verify positivity of the rest

Bernstein Polynomials

$$B_{n,i,a,b} = \binom{n}{i} \frac{(X-a)^i (b-X)^{n-i}}{(b-a)^n}$$

- Proportional to $\mu_{n,a,b}^{-1}(X^i)$
- Coefficients in Bernstein basis have same sign
- Coefficients in Bernstein basis have a geometric interpretation

Bernstein Polynomial of degree 5

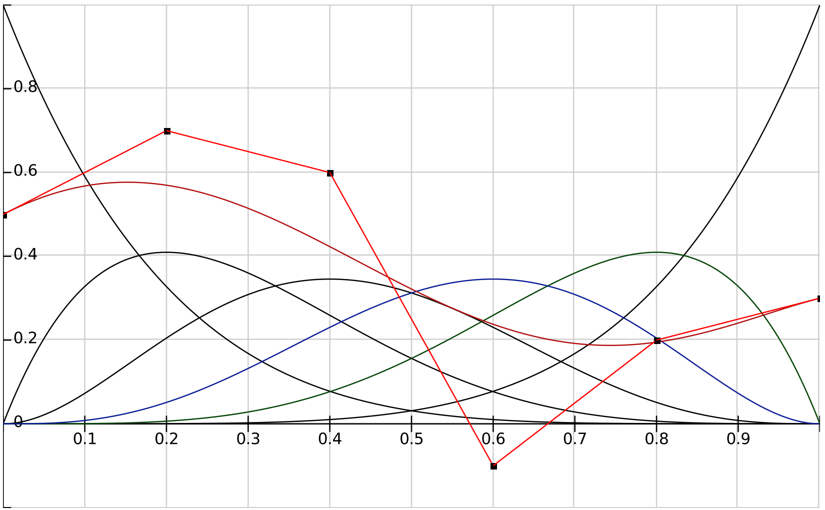


Bernstein Control Points

$$P = \sum_{i=0}^{n} b_i B_{n,i,a,b}$$

- consider the points $(a + i\frac{(b-a)}{n}, b_i)$
- The broken line that links them approximates the curve
- The convex hull of these points contains the curve

Bernstein Control Points

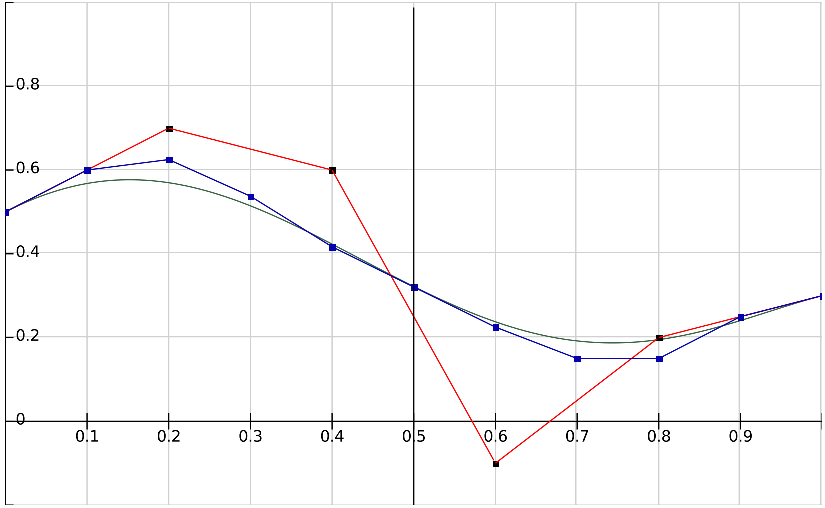


http://fooplot.com/plot/khmozg13qz

Dichotomy

- If all coefficients are positive, this is sufficient
- If one of the coefficients is negative, no conclusion
- Solution: compute Bernstein coefficients for a smaller interval

Dichotomy



http://fooplot.com/plot/qilqpfevwd

Implementation

- Leveraging existing tactics
- Datatypes for fractional expressions `Fexpr` and polynomial expressions `PExpr`
  - Constructors for addition, multiplication, variables, opposite, constants (integers)
- A datatype for normalized polynomial expressions, `Pol`
  - Like list of coefficients but more efficient (cater sparsity)
- Normalization from one type to a different type
  - Easy to collect "coefficients"
  - Problem when normalization is a middle step

Basic operations

- Composition with a polynomial expression: substitution
  - Replace a variable by a `FExpr` in a `FExpr`
  - Replace a variable by a `PExpr` in a `PExpr`
- reversing a polynomial: directly on normalized polynomials
- Added a function from `Pol` to `PExpr`

Use in a multi-variate context

- The tactic takes as an input a multi-variate positivity goal and a *principal* variable
- Produces a collection of multi-variate positivity goals
  - The principal variable has disappeared from goals
- Companion tactics make it possible to perform dichotomy

Demo time

conclusion

- Dichotomy can be performed directly on coefficients
  - Casteljau's algorithm, similar to Pascal triangle
  - Proof of correctness already performed (in Coq): Bertot,Guilhot&Mahboubi2010
- Complexity needs to be tamed
- No reflective implementation yet
- Certificate approach is relevant: explain the tree of dichotomies
  - Wish to re-use existing work: Solovyev (Flyspeck), Zumkeller

Computing dichotomies

0.5		0.7		0.6		−0.1		0.2		0.3
	0.6		0.65		0.25		0.05		0.25	
		0.625		0.45		0.15		0.15		
			0.5375		0.3		0.15			
				0.41675		0.225				
					0.32075					

- Here, half-sums
- Also possible to use pondered averages $(\alpha x + (1 - \alpha) y)$
  - α may be outside $(0, 1)$