Bernstein Coefficients

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Basic idea

- Proving positivity of a polynomial in a bounded interval
- Two points of view
  - Variable changes to change the nature of intervals
  - Basis change in the polynomial vector space
- Obtain a sufficient positivity condition on the coefficients
- A natural approach for eliminating a variable
- Condition is not necessary
  - Incompleteness is solvable thanks to dichotomy
Variable changes

- Positivity of $P$ on interval $(a, b)$ equivalent to positivity of $P \circ (X \ast (b - a) + a)$ on interval $(0, 1)$
- Positivity of $P$ on interval $(a, +\infty)$ equivalent to positivity of $P \circ (X + a)$ on $(0, +\infty)$
- Positivity of $P$ on $(0, 1)$ equivalent to positivity of $X^n \times P(1/X)$ on $(1, +\infty)$
  - Last operation actually stays in the same polynomial space (if $\text{deg}(P) = n$)
  - Alternative point of view: *reversing the list coefficient*
  - A linear operation
Linear transforms

- for any \( Q, P \mapsto P \circ Q \) is linear
- call \( \theta_a : P \mapsto P \circ (X + a) \)
- call \( \chi_a : P \mapsto P \circ (aX) \)
- call \( \rho_n : \sum_{i=0}^{n} c_i X^i \mapsto \sum_{i=0}^{n} c_{n-i} X^i \)
- positivity of \( P \) on interval \((a, b)\) equivalent to positivity of \( \theta_a \circ \chi_{b-a}(P) \) on \((0, 1)\)
- positivity of \( P \) on interval \((0, 1)\) equivalent to positivity of \( \theta_1 \circ \rho_n(P) \) on \((0, +\infty)\)
Basis change

- The operation $\mu_{n,a,b} = \theta_1 \circ \rho_n \circ \theta_a \circ \chi_{b-a}$ is linear, invertible.
  - On vector space of polynomials of degree $\leq n$
- Coefficients of $\mu_{n,a,b}(P)$ in monomial basis $(1, X, \ldots, X^n)$ are coefficients of $P$ in basis $\mu_{n,a,b}^{-1}(1, X, \ldots, X^n)$

$$\mu_{n,a,b}(P) = \sum_{i=0}^{n} b_i X^i \iff P = \sum_{i=0}^{n} b_i \mu_{n,a,b}^{-1}(X^i)$$

- $\forall i, 0 < b_i \Rightarrow \forall x, a < x < b \Rightarrow 0 < P(x)$
- $\mu_{n,a,b}^{-1}(X^k) = \frac{(X-a)^{n-k}(b-X)^k}{(b-a)^n}$
- The polynomials $\mu_{n,a,b}^{-1}(X^k)$ are obviously positive on $(a, b)$
Proof procedure

- Use $\mu_{n,a,b}$ to compute coefficients $b_i$
- Verify the equality

$$P = \sum_{i=0}^{n} b_i \frac{(X - a)^{n-k}(b - X)^k}{(b - a)^n}$$

- get rid of null coefficients, and verify positivity of the rest
Bernstein Polynomials

\[ B_{n,i,a,b} = \binom{n}{i} \frac{(X - a)^i (b - X)^{n-i}}{(b - a)^n} \]

- Proportional to \( \mu_{n,a,b}^{-1}(X^i) \)
- Coefficients in Bernstein basis have same sign
- Coefficients in Bernstein basis have a geometric interpretation
Bernstein Polynomial of degree 5
Bernstein Control Points

\[ P = \sum_{i=0}^{n} b_i B_{n,i,a,b} \]

- consider the points \((a + i\frac{(b-a)}{n}, b_i)\)
- The broken line that links them approximates the curve
- The convex hull of these points contains the curve
Dichotomy

- If all coefficients are positive, this is sufficient
- If one of the coefficients is negative, no conclusion
- Solution: compute Bernstein coefficients for a smaller interval
Dichotomy

http://fooplot.com/plot/qilqpfewvwd
Implementation

- Leveraging existing tactics
- Datatypes for fractional expressions $\text{FExpr}$ and polynomial expressions $\text{PExpr}$
  - Constructors for addition, multiplication, variables, opposite, constants (integers)
- A datatype for normalized polynomial expressions, $\text{Po1}$
  - Like list of coefficients but more efficient (cater sparsity)
- Normalization from one type to a different type
  - Easy to collect “coefficients”
  - Problem when normalization is a middle step
Basic operations

- Composition with a polynomial expression: substitution
  - Replace a variable by a FExpr in a FExpr
  - Replace a variable by a PExpr in a PExpr

- Reversing a polynomial: directly on normalized polynomials

- Added a function from Pol to PExpr
Use in a multi-variate context

- The tactic takes as an input a multi-variate positivity goal and a *principal* variable
- Produces a collection of multi-variate positivity goals
  - The principal variable has disappeared from goals
- Companion tactics make it possible to perform dichotomy
Demo time
Dichotomy can be performed directly on coefficients
  ▶ Casteljau’s algorithm, similar to Pascal triangle
  ▶ Proof of correctness already performed (in Coq): Bertot, Guilhot & Mahboubi 2010

Complexity needs to be tamed

No reflective implementation yet

Certificate approach is relevant: explain the tree of dichotomies
  ▶ Wish to re-use existing work: Solovyev (Flyspeck), Zumkeller
Computing dichotomies

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- Here, half-sums
- Also possible to use pondered averages \((\alpha x + (1 - \alpha)y)\)
  - \(\alpha\) may be outside \((0, 1)\)