Interval computations and some computable analysis

Florian Steinberg (some is joint work with Laurent Théry)

June 21, 2018

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Formalize results from computable analysis.

Chebyshev Approximation

Interval computation

Some computable analysis

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- T_m is a basis $\rightsquigarrow$ exist uniqe a_i s.t. $p = \sum_{i \leq \deg(p)} a_i T_m$.

Analytic facts

Alternate trigonometrical Definition:

$$T_n(x) = \cos(n \operatorname{acos} x), \text{ if } x \leq 1.$$

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$$\int_{-1}^{1} \frac{T_n(x)\, T_m(x)}{\sqrt{1-x^2}}\, dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = 0 = m \\ \frac{\pi}{2} & \text{if } n = m \neq 0 \end{cases}$$

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$\rightsquigarrow$ closely related to Fourier-series.

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- Proved $T_n(x) = \cos(n \operatorname{acos}(x))$.
- Proved orthogonality.

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An interval extension of $f : \mathbb{R} \to \mathbb{R}$ is a function $F : \mathbb{ID} \to \mathbb{ID}$ such that:

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replace operations in the definition of f by Interval operations.

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match l with
| [::]  => 0
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Prove correctness by means of interval enclosures:

Lemma (hornerIA_correct)

```
∀l x lI I:
(∀i, l_i contained_in lI_i) ->
x contained_in I ->
p.[x] contained_in (hornerIA pI I).
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Clenshaw algorithm with a proof of correctness.

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More interesting: $\mathbb{R}$.

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Example (The Cauchy rep. δ_C)

Set $Q := \mathbb{Q}$ and $A := \mathbb{Q}$. $\varphi\colon \mathbb{Q} \to \mathbb{Q}$ is δ_C-name of $x \in \mathbb{R}$ iff

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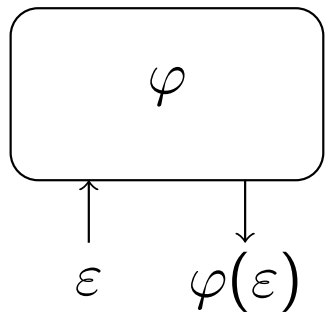

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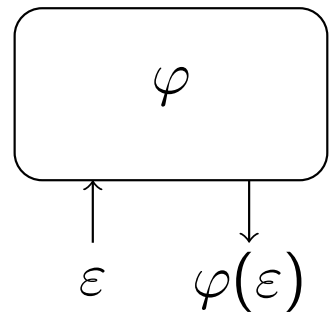

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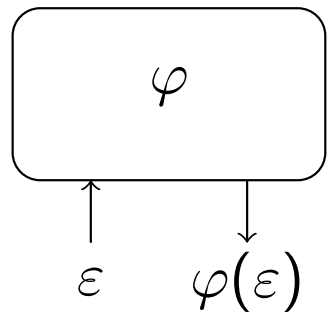

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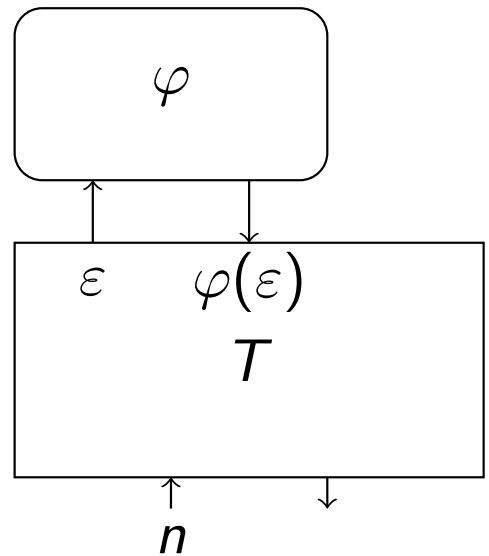

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Monotone sequence I_n of intervals is a δ_i-name of $x \in \mathbb{R}$ if

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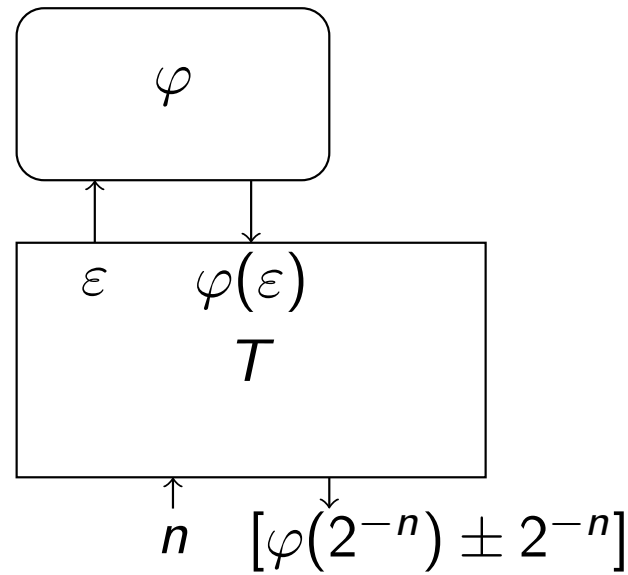

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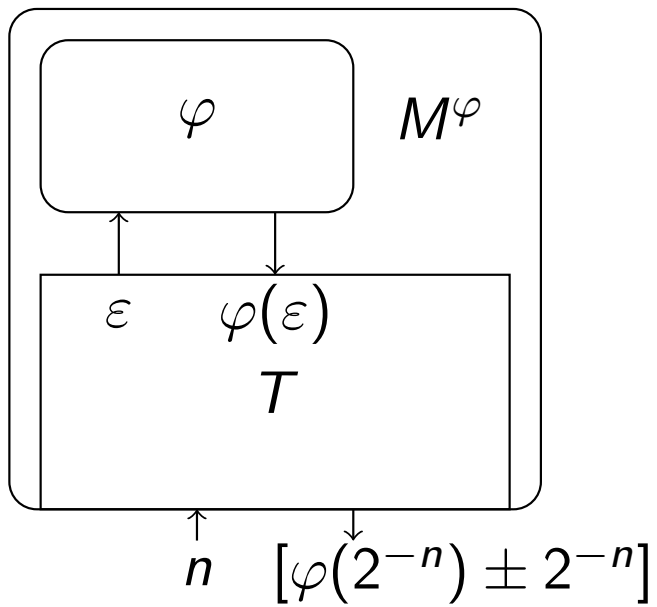

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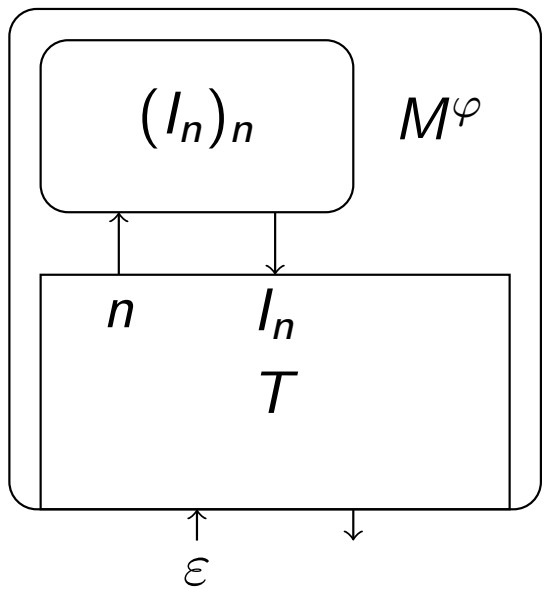

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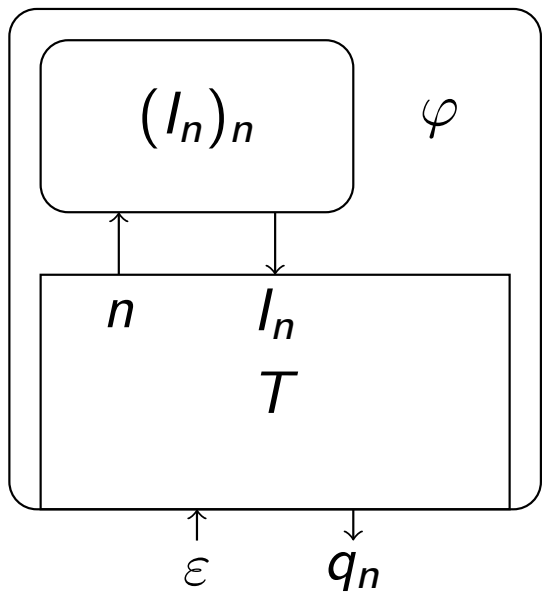
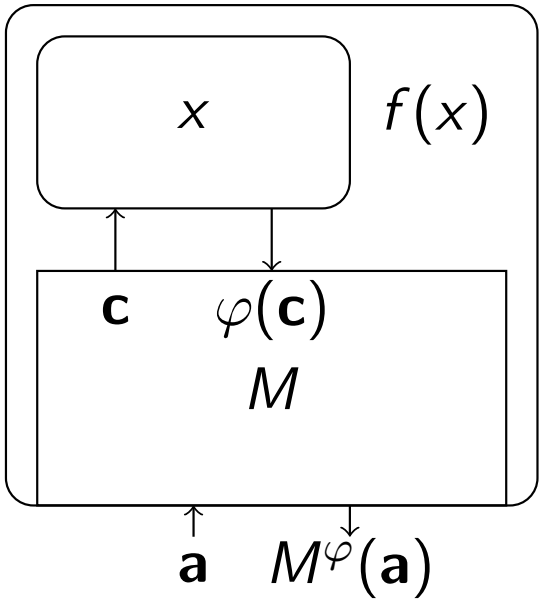


Functions on the interval reals



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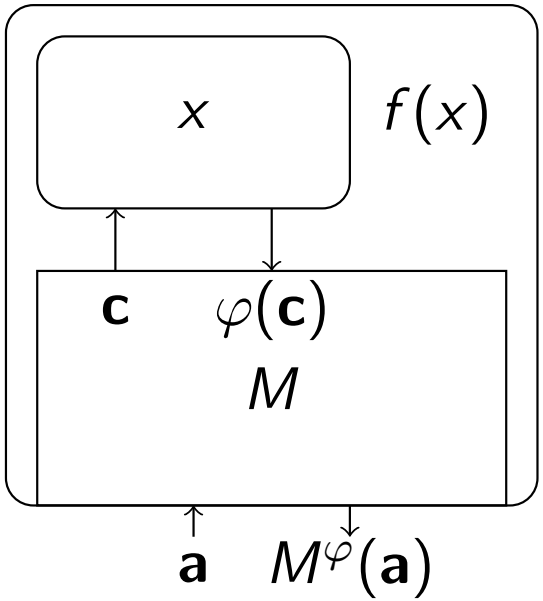

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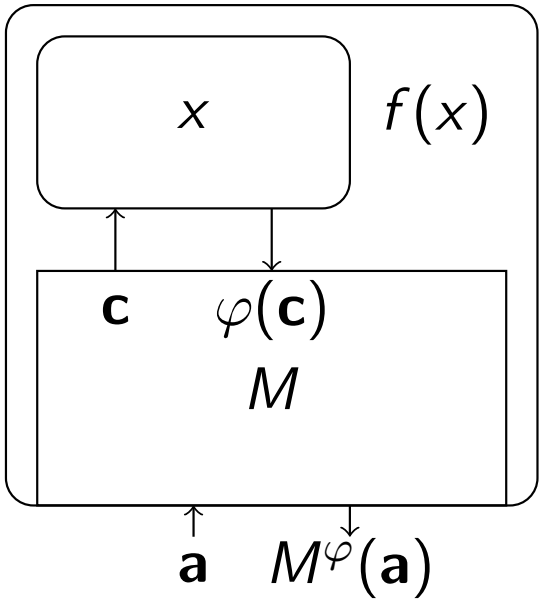

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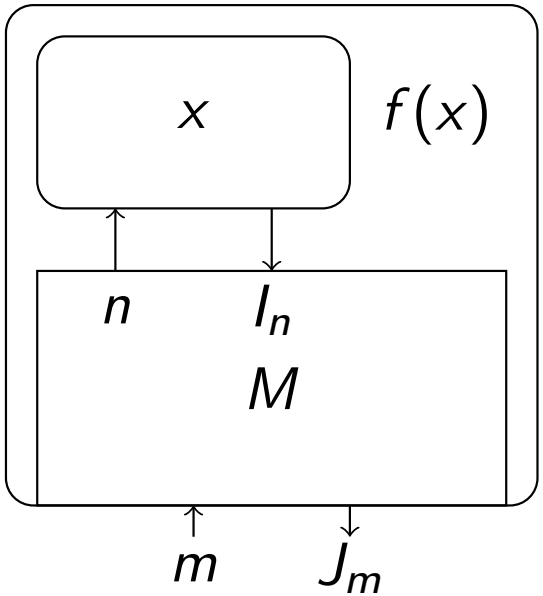

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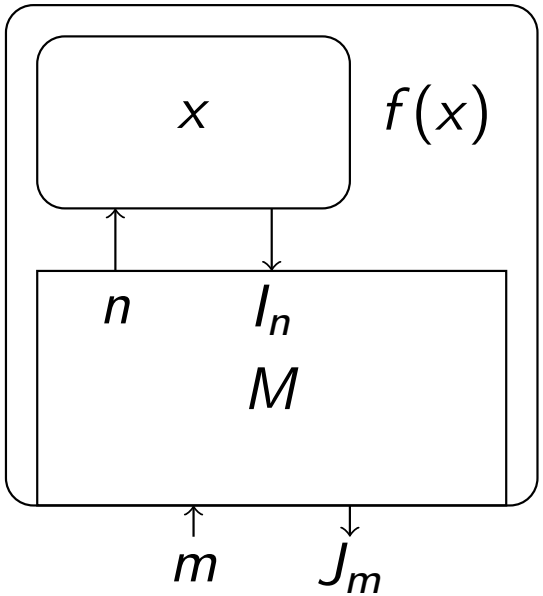

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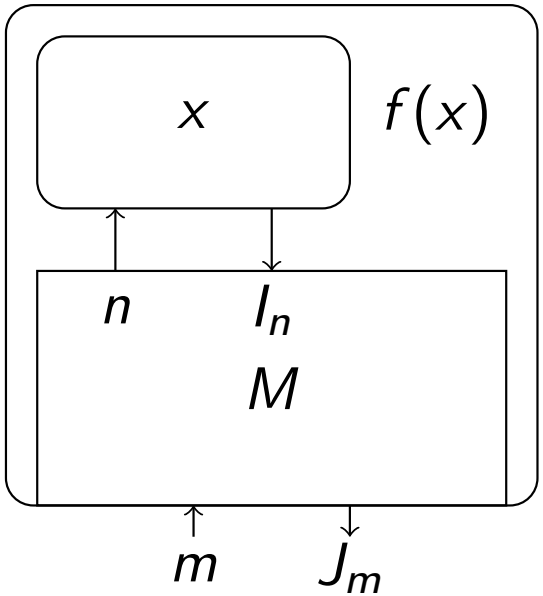

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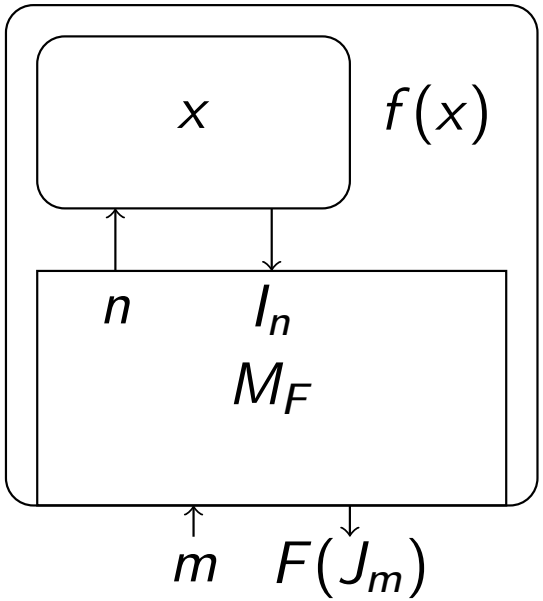

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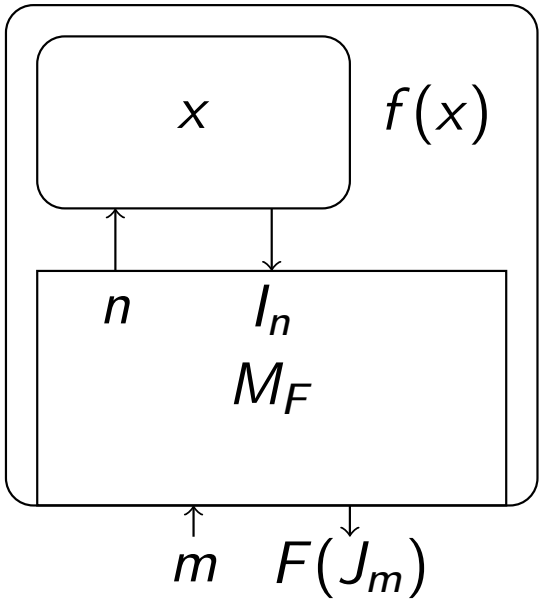

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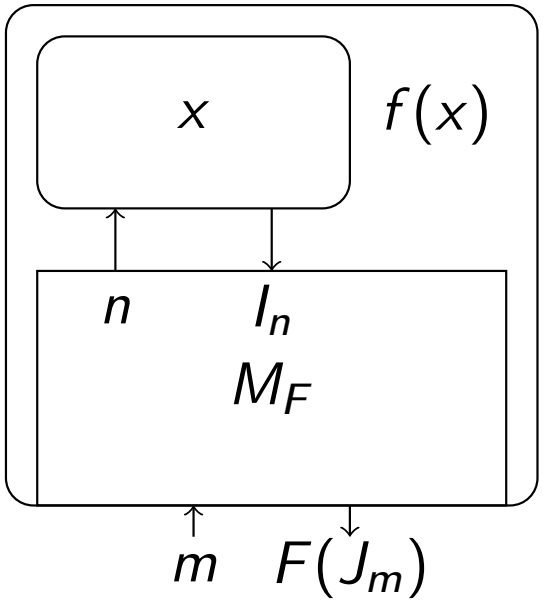

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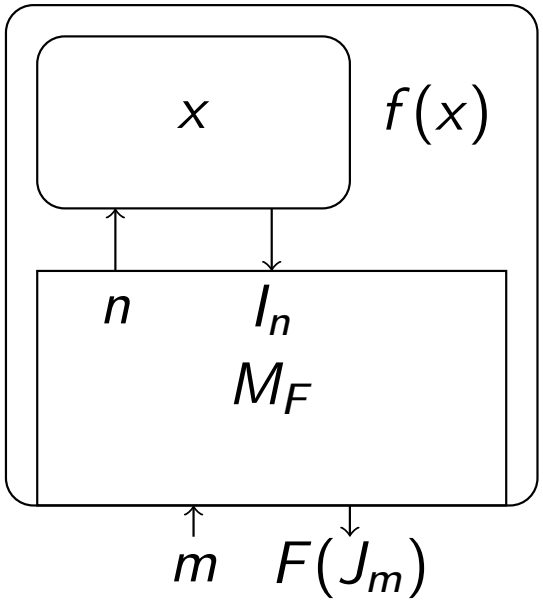

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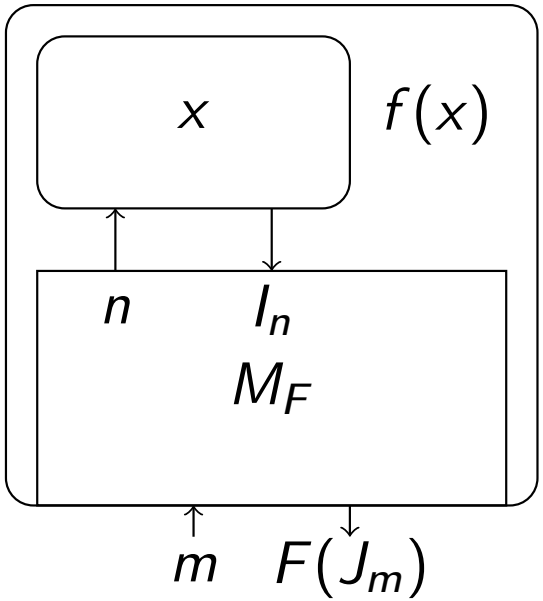

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- Computable analysis uses different concept of computability.

Thanks!

Thanks!

On the Github:
`https://github.com/FlorianSteinberg/Cheby`
`https://github.com/FlorianSteinberg/coqrep`