Semi-Automatic Real Asymptotics in Isabelle/HOL

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- Archive of Formal Proofs:
  Large collection of Isabelle proof developments

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$$\lim_{x\to\infty} \left(1 - \frac{1}{b\log^{1+\varepsilon} x}\right)^p \left(1 + \frac{1}{\log^{\varepsilon/2}\left(bx + \frac{x}{\log^{1+\varepsilon} x}\right)}\right) - \left(1 + \frac{1}{\log^{\varepsilon/2} x}\right) \;=\; 0^+$$

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In Isabelle: 700 lines of messy proofs

lemma akra_bazzi_aux:
filterlim
$(\lambda x.\ (1 - 1/(b * \ln x\ \hat{}\ (1 + \varepsilon))\ \hat{}\ p) *$
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Computer Algebra Systems can do this (sort of)

So why can't we?

Multiseries Expansions

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Related Work:

- *Asymptotic Expansions of exp–log Functions*
  by Richardson, Salvy, Shackell, van der Hoeven
- *On Computing Limits in a Symbolic Manipulation System*
  by Gruntz

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- The basis must be ordered descendingly by 'growth class': $\forall i.\ \ln b_{i+1}(x) \in o(\ln b_i(x))$
- Typical basis: $\exp(x^2)$, $\exp(x)$, x, $\ln x$, $\ln \ln x$

A coalgebraic view of Multiseries

type Basis $=$ $(\mathbb{R} \to \mathbb{R})$ list

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\textbf{type}\ \mathsf{Basis}\ =\ (\mathbb{R} \to \mathbb{R})\ \mathsf{list} \\
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- exp and ln at singular points require specialised procedures and may add new basis elements
- For operations like Γ, erf, li:
  factor out singularities and treat them separately

Connecting Series and Functions

For simple power series, $f \sim ts$ can be expressed coinductively:

$$\frac{f(x) \in O(x^e) \qquad f(x) - c\,x^e \;\sim\; ts}{f(x) \;\sim\; (c, e) :: ts}$$

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Similar for multiseries.

Finding expansions

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which evaluates to

$$\exp(x) + x^{-1} - \frac{1}{6}x^{-3} + \frac{1}{120}x^{-5} - \ldots$$

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This works surprisingly well

Proof method

With some pre-processing, we can automatically prove statements of the form

- $f(x) \longrightarrow c$
- $f(x) \sim g(x)$
- $f(x) \in L(g(x))$ for any Landau symbol L

as $x \to l$ for $l \in \mathbb{R} \cup \{\pm\infty\}$

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sin, cos, tan at finite points also possible.

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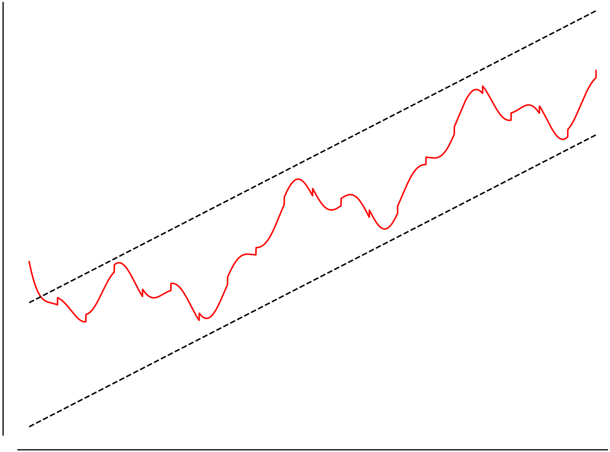

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  by (real_asymp simp: mult_neg_pos)

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Maxima takes very long for some of them and gives wrong result for this:

$$\exp\left(\frac{\log\log\left(x+e^{\log x \log\log x}\right)}{\log\log\log\left(e^x+x+\ln x\right)}\right) \longrightarrow e$$

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Isabelle still isn't a CAS – but we're getting there.

Questions? Demo?