Distant decimals of π

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- Arithmetic-geometric means: an algorithm with 3n divisions
- Arithmetic-geometric means: an algorithm with only one division
- A few facts about the correctness of individual digits
- The BBP formula

The Arithmetic-Geometric computation

- A pair of two sequences of real numbers
- $a_0 = 1 \quad b_0 = x \quad a_{n+1} = \frac{a_n + b_n}{2} \quad b_{n+1} = \sqrt{a_n b_n}$
- $\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{(1^2+t^2)(x^2+t^2)}} = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{(a_n^2+t^2)(b_n^2+t^2)}}$
- a_n and b_n converge very fast to the same value (a_n is larger)
- $a_n - b_n < A^{-2^n}$
- Number of known digits doubles at every step

AGM for π

- Define $f(x)$ as the arithmetic-geometric mean of 1 and x
- Key property $\pi = 2\sqrt{2}\frac{f(1/\sqrt{2})^3}{f'(1/\sqrt{2})}$
- Mathematical proofs based on elliptic integrals
  - Improper integrals (Coquelicot)
  - Needed extensions (Chasles with improper integrals)
  - Lots of formulas with squares and square roots slightly beyond Presburger automated proofs

Borwein&Borwein algorithm

- Define $y_n(x) = \dfrac{a_n(x)}{b_n(x)} \qquad z_n(x) = \dfrac{b_n'(x)}{a_n'(x)}$
- Easy consequence $1 + y_n = 2\dfrac{a_{n+1} b_{n+1}^2}{a_n b_n^2} \qquad 1 + z_n = 2\dfrac{a_{n+1}'}{a_n'}$

$$y_0(x) = \frac{1}{x} \quad y_{n+1} = \frac{1 + y_n}{2\sqrt{y_n}} \quad z_1 = \frac{1}{\sqrt{x}} \quad z_{n+1} = \frac{1 + z_n y_n}{(1 + z_n)\sqrt{y_n}}$$

- $\pi_n = (2 + \sqrt{2}) \displaystyle\prod_{i=1}^{n} \frac{1 + y_i}{1 + z_i}$ at $x = \dfrac{1}{\sqrt{2}}$

convergence rate

$$0 \leq \pi_{n+1} - \pi \leq 8\sqrt{2} \times 531^{-2^n}$$

Brent and Salamin algorithm

- Define $c_n = \frac{1}{2}(a_{n-1} - b_{n-1})$
- $\dfrac{b'_{n+1}}{b_{n+1}} - \dfrac{b'_n}{b_n} = \dfrac{b_n}{2a_n}\left(\dfrac{a_n}{b_n}\right)' \qquad \dfrac{b'_{n+1}}{b_{n+1}} = \dfrac{b'_1}{b_1} - \sqrt{2}\sum_{k=1}^{n-1} 2^k c_k^2$
- $\pi'_n = \dfrac{4a_n^2}{1 - \sum_{k=1}^{n-1} 2^{k-1}(a_{k-1} - b_{k-1})^2}$

convergence rate

- First approximation $|\pi'_{n+1} - \pi| \leq 68 \times 531^{-2^{n-1}}$
- Coarser than Borwein&Borwein: $\pi_{n+1} - \pi \leq 8\sqrt{2} \times 531^{-2^n}$
- Improvement by studying $|\pi'_{n+2} - \pi'_{n+1}|$

$$|\pi'_{n+1} - \pi| \leq (132 + 384 \times 2^n) \times 531^{-2^n}$$

- For computing 10^6 decimals of π both π'_{19} and π_{19} are enough
  - $132 + 384 \times 2^{19} \leq 2^{28}$
- Each algorithm computes n square roots, but π_n computes $3n$ divisions, π'_n only half-sums and one full division.

Rounding errors

- Approximating real computations using fixed-point computations (rounding towards 0)
- Take the same program code, replace operations
- How do rounding errors propagate?
- Amazingly $|\boxed{y_n} - y_n| \leq 2ulp$ and $|\boxed{z_n} - z_n| \leq 4ulp$
- $|\boxed{\pi_n} - \pi_n| \leq (21 * n + 2)ulp$
- $|\boxed{\pi'_n} - \pi'_n| \leq (160(\frac{3}{2})^{n+1} + 80 * 3^{n+1} + 100)ulp$

Total interval for Brent-Salamin

- In the end $\boxed{10^{10^6+k} \times \pi'_{19}}$ is within 2^{40} of $10^{10^6+k} \times \pi$
- It remains to choose a suitable value of k
- It is more efficient to compute $\boxed{2^{\lfloor log_2(10) \rfloor \times 10^6} \times \pi'_{19}}$
- Paper available at `https://hal.inria.fr/hal-01582524`
- Code and instructions `https://www-sop.inria.fr/marelle/distant-decimals-pi/`
- Includes a C implementation of the Borwein algorithm, on top of MPFR.

The BBP formula

Work done in Coq by Laurence Rideau and Laurent Théry

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right).$$

- Mathematical justification using Rieman integrals
- commuting integration and infinite sum: thanks to Coquelicot

Computation of a single (hexa-decimal) digit

- Computing the d^{th} hexadecimal digit of π
- $\lfloor 16^{d-1}\pi \rfloor (mod 16)$
- Choose a precision p
- Compute separately each of the sums, taking the modulo right-away
- Example: $\sum_{i=0}^{\infty} \lfloor \frac{2^p 16^{d-1-i} \times 4}{8i+1} \rfloor (mod 2^p)$

Intuitive example

- Transposing in base 10, what is the second digit (on the right of the dot) of
$$\sum_{i=0}^{\infty} \frac{1}{10^i} \frac{1}{2i+1}$$
- $1 + 0.03333\ldots + 0.00200\ldots + 0.00014\ldots + 0.00001\ldots + \cdots$

Computation of a single digit (continued)

- Only a finite number of terms in the sum, approximately $d + p/4$ terms
  - No modulo is needed for the last $p/4$ terms
- Use integer division: uncertainty bounded by 1
- Accumulated uncertainty is $d + p/4 + 1$
- We need the accumulated uncertainty to be small wrt. 2^{p-4}
- Final result is partial: when the value modulo 2^p is closer to 2^p than the accumulated uncertainty the digit is not known.